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Sensitive Dependence to Parameters, Fat Fractals,

and Universal Strange Attractors

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There are many nonlinear differential equations for which two different types of behavior, such as chaos and periodicity, are interwoven in a complex and intricate manner, so that the bifurcation parameters form a "fat fractal". The result is that statistical averages vary wildly with parameters and, strictly speaking, prediction becomes impossible even in the statistical sense. (For example, climate, as well as weather, is unpredictable.) There is, however, order in this unpredictable behavior, which can be described by a universal strange attractor of the renormalization transformation.

Introduction

Differential equations and their discrete counterparts, mappings, occur in almost every branch of science. The generic name for this type of mathematical model is *dynamical system*. Loosely put, this is nothing more than a rule stating how some quantity, or set of quantities change, usually with time. A familiar example is Newton's law; $\vec{F} = m\ \vec{a}$ describes how the position of an object varies in time. The force $\vec{F}$ gives the rule, the acceleration $\vec{a}$ relates this rule to position, and the mass m is what is called a *parameter* of the equation, i.e. something that stays constant for any given application, but might vary from case to case. Dynamical systems can be divided into two broad categories, linear and nonlinear. Although most physical problems are most accurately modeled by nonlinear equations, linear models have been more commonly utilized. The reason is expedience; linear systems can be put in a form in which each variable behaves independently of the others, and are thus solvable. In contrast, there is no systematic theory for solving nonlinear systems.

Even though it is generally not possible to solve nonlinear equations, it is possible to simulate their behavior on a computer, and by doing this in recent years considerable progress has been made toward understanding their qualitative properties. In very general terms, the results of some of these investigations can be summarized as follows:

First, the properties of nonlinear equations are dramatically different than those of linear equations. In particular, they are capable of several different varieties of "sensitive" behavior. The most famous of these is *chaos,* also called *sensitive dependence to initial conditions,* conjectured by Poincare [1] at the turn of the century, and developed by Lorenz [2] in the early 60's. Chaos has received a great deal of attention in recent years, since it explains how chaotic, apparently random behavior can be generated by a physical system following deterministic laws. The basic discovery is that for some nonlinear dynamical systems errors in initial measurements grow geometrically, rather than arithmetically in time. Small changes in initial values produce very large changes at a later time, and the detailed behavior of the system becomes unpredictable in anything other than a statistical sense. Chaos is believed to be the underlying cause of many different phenomena that seem to contain random or unpredictable elements, such as the weather. It puts an inherent limit on our ability to predict the future, since with initial measurements of only finite accuracy, it is impossible to predict the details of future behavior.

The long-time behavior of a dynamical system, after "transients" have been allowed to die out, is often more important than the sort-term behavior. In dynamical systems with some form of friction, or dissipation, initial conditions are "attracted" to some subset of all possible values, called an *attractor.* For example, the motion of a mass on the end of a spring will eventually damp out, approaching a state of rest called a fixed point attractor. A nonlinear oscillator (with an energy source) can have more complicated attractors, such as a limit cycle attractor; this means that after all the transients die out, motion always approaches a periodic cycle that is the same regardless of the particular initial conditions. A metronome, the heart, or the feedback produced when a microphone is held up to a speaker are all examples of limit cycle attractors. A more complicated kind of attractor, called a *strange attractor,* appears when motion on an attractor is chaotic. This is a good example of the concurrence of order and chaos; the attractor represents a restriction of the motion, a reduction of possiblities; within the restrcitions imposed by the attractor, however, motion is chaotic.

A second result that was quite surprising is the existence of *universal properties* of nonlinear equations. Even though there are an infinite number of different nonlinear equations, many of them behave in essentially the same wav. Perhaps the first instance of this was discovered by Metropolis, Stein, and Stein [3] in the early seventies. on. An aspect of this work was substantially expanded by Feigenbaum [4], who showed that one of the most common ways to make a transition from predictable to chaotic behavior always occurs in exactly the same way. Universality implies that even though the behavior of nonlinear equations may be quite complex, there are orderly patterns to the way in which this complex behavior occurs, common to all equations in a given class.

Sensitive Dependence to Parameters

The central purpose of the work outlined here is to discuss a lesser known "sensitive" property of nonlinear systems, called *sensitive dependence to parameters,* and to demonstrate some of its universal properties. Roughly put, sensitive dependence to parameters occurs when a dynamical system's behavior changes wildly as a parameter (such as the mass m in $\vec{F} = m\ \vec{a}$) is varied. Thus, for example, at one parameter value a system might be chaotic, at another nearby value periodic, and then again chaotic, etc. The remarkable aspect is that arbitrarily close to every parameter value where there is chaos, there is another parameter value where there is a stable periodic orbit. At the same time, a finite fraction of parameter values generate chaos. In fact, the parameter values where the behavior is changing, or the *bifurcation parameters,* consume a finite fraction of all possible parameter values, equal to the set of chaotic parameter values. The practical implications for prediction are bad, worse even than for chaos: For chaos, the details of future behavior are unpredictable, but for sensitive dependence to parameters, prediction is not even possible in a statistical sense, since the average behavior when the system is chaotic may be completely different from its behavior when it is periodic.

This phenomenon was originally recognized by Edward Lorenz, a meteorologist, who described it in a little known paper called "On Determining the Climate from the Governing Equations" [5]. The basic point of this paper is that the climate, which is an average or statistical property of the equations that govern the weather, may be inherently unpredictable due to sensitive dependence to parameters. A discussion of the example used by Lorenz will perhaps help to make this clearer.

Since the equations that really describe the weather are quite complicated, and extremely difficult to deal with, Lorenz chose to study a very simple nonlinear equation, called the "logistic map", which may thought of as a metaphor for the weather.

$$x_{k+1} = r x_k (1 - x_k). \tag{1}$$

Here x is a number between zero and one, r is a parameter value between zero and four, and k is a label that plays the role of time. For any given initial value x_0 and a fixed value of r this equation can be used to generate a new value x_1, which can in turn

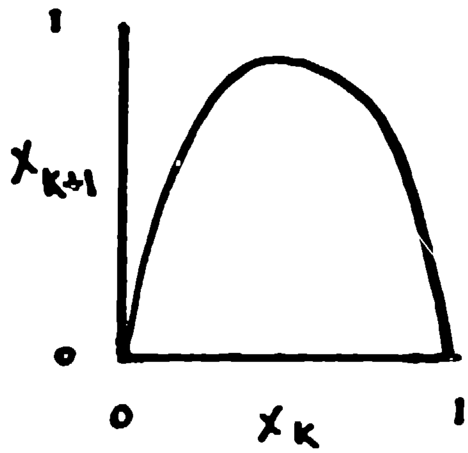

Figure 1.
An example of a very simple nonlinear dynamical system, given by Eq (1).

be used to generate x_2, *ad infinitum*. The sequence x_k might describe some property of the weather, such as the temperature on successive days, and r might describe a fixed property, such as the latitude. Of course, this equation is much too simple for a realistic model of the weather, but Lorenz's purpose was to provide an illustration; if this very, very simple equation can do unpredictable, complex things, then complicated equations such as those that actually underly the weather might also be capable of the same type of behavior.

Lorenz demonstrated that there are many values of r where the sequence x_k is chaotic, i.e., the values of x_k appear to hop around in a random manner, never settling down. At other values of r, the sequence x_k is asymptotically periodic, or in other words, the values of x_k eventually begin repeating themselves, settling down into an unvarying pattern in which x periodically changes between fixed values (a limit cycle attractor). Lorenz demonstrated that arbitrarily close to any value of r generating chaos, there is another value of r generating periodicity. Furthermore, for chaotic parameter values the average value of x may take on one value, while at the arbitrarily nearby periodic values, the average may be quite different. The result is that unless r is known with infinite accuracy (which it never is), the average value of x is unpredictable, since even an infinitesmal change in r might produce a substantial change in the average. If x represents temperature, then the average temperature would be unpredictable. Identifying the average temperature as a feature of climate, we see that the the climate in this case is unpredictable *in principle*.

Fat Fractals

The fact that there are periodic parameter values arbitrarily close to chaotic parameter values has led many people to assume that chaos must be unlikely to occur in such equations, and that the erratic behavior seen in computer experiments must be an artifact of the computer errors. This is in fact wrong. As was originally demonstrated by Lorenz [5], and lately proved by Jakobsen [6], the chaotic parameter values consume a finite fraction of the values of r. To better understand how this seemingly paradoxical behavior occurs, let us first construct a hypothetical set with analogous properties: Delete the middle third of a line segment extending from 0 to 1, leaving two line segments. Now delete the middle third of each of these, leaving four segments. Continue this process *ad infinitum,* as shown in Figure 2. What is left is called a Cantor set, and has some very remarkable properties. First, there is a gap arbitrarily close to any point in the Cantor set. This implies that the Cantor set is its own boundary. Second, the points in the Cantor set cannot be counted, even though when "added up", their total length is zero. In other words, picking a point at random, the probability that it lies in the Cantor set of (a) is zero. Thus, this can be called a "thin" Cantor set.

It is possible to "fatten" this Cantor set by changing the scaling between the gaps, as shown in Fig. 2(b). At the second stage of construction, delete the middle 1/9 rather than the middle third of each segment, and at the third stage delete the middle 1/27 of each segment, and so on. The result is that this new "fat" Cantor set has length greater than zero; if a point is picked at random, there is a nonzero chance that it will be one of the points in the fat Cantor set. This is true even though there are still holes arbitrarily close to any point in the fat Cantor set, so that the fat Cantor set is still its own

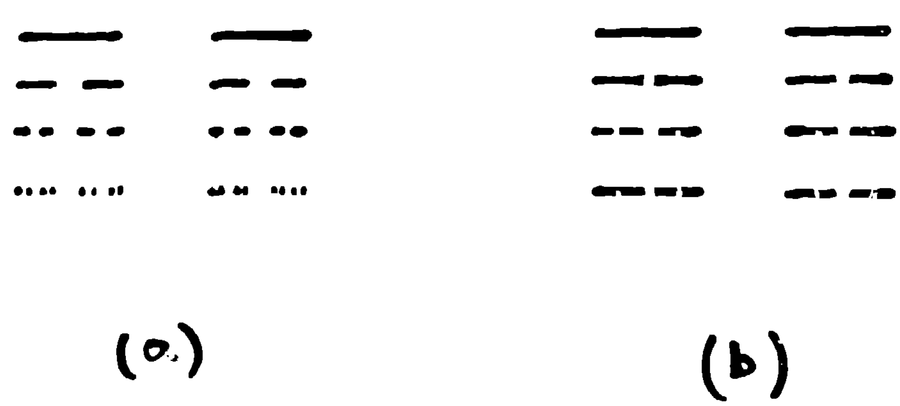

Figure 2.
Two examples of Cantor sets, as described in the text. The one shown in (a) is "thin", and the one shown in (b) is "fat".

boundary.

The set of parameter values where chaos occurs in the logistic map is also a fat Cantor set; think of the holes as periodic values of r, and the points in the fat Cantor set as the chaotic values. The set of bifurcation parameters is just the set of boundary points, and is equal to the set of chaotic values. Thus every parameter value causing chaos is also a bifurcation parameter. Note: Since a Cantor set is a very specific thing, the more general term *fractal* [7] is often used to discuss objects of this type. Like Cantor sets, fractals can be either fat or thin.

In examining a fat fractal such as the one of Figure 2, even though there are gaps arbitrarily close to every point, only a few of them are visible. The pen used to draw the figure has a finite width, and in any case your eyes have limited resolution, so that the smallest gaps are obliterated. The size of the fat fractal that you actually see is larger than the true size. If somehow the pen were finer and your eyes were better, more of the gaps would become visible, and the apparent size would decrease to reach a value closer to the true value. The apparent size thus depends on the scale of resolution.

The same is true of the set of chaotic parameter values of the logistic map. One of the new results being reported here [8] is that the the apparent size of fat fractals of this type changes as

$$L(\epsilon) = L(0) + k\,\epsilon^{\beta} \qquad (2)$$

where $L(\epsilon)$ is the total length measured using resolution ϵ, and k and β are constants. (This relation is only generally valid for small ϵ.) The exponent β provides a means of quantifying the extent to which something is a fat fractal. If β is close to zero, then the area only changes slowly with changes in the resolution, and the fractal property is very strong, whereas if β is large, then the fractal property is weak. β can in fact be used to *define* sensitive dependence to parameters. Specifically, sensitive to parameters occurs when when $\beta < \infty$, i.e., when there is a fat fractal in parameter space. This same property has recently been shown to hold for other types of equations as well [9].

Thus far, we have been assuming that the motion we are discussing is fully deterministic, i.e., there are no external random influences acting on the system. In real physical problems, however, there are always fluctuating, unknown, apparently random external influences at work. Even though these effects may be very small, in the case of sensitive dependence to parameters, they play an important conceptual role. In particular, as demonstrated in reference [10], random external fluctuations wipe out all the stable periodic orbits below a certain minimum size, depending on the amplitude of the external influences.) The result is that external random fluctuations actually make it possible to predict statistical averages, since they smooth out all the complicated behavior associated with sensitive dependence to parameters.

The complication introduced by sensitive dependence to parameters, then, is that the phenomena observed depend in an essential way on the level of the external fluctations. In the presence of sensitive dependence to parameters two experiments done with different levels of external fluctuations will observe different results, with the less noisy experiment observing more structure and more bifurcations than the other. Strictly speaking, neither of them is "right", since another experiment with an even lower level of fluctuations will always observe more structure. The amount of new behavior that emerges as the level of fluctuations is lowered can be predicted from knowledge of the exponent β of Equation (2). Thus β provides a means of summarizing the effect of external fluctuations on sensitive dependence to parameters. More detailed predictions can be made from the properties of the appropriate universal strange attractor, discussed below.

Universal Strange Attractors

In spite of all this pessimism concerning predictablity, the logistic equation, and all other equations of the same type, have some very orderly properties, one of which discovered by Metropolis, Stein, and Stein [2]. In particular, the stable periodic orbits discussed above can be labelled according to whether each step of the orbit goes to the left or right of $x = 1/2$. ($x = 1/2$ is called the *critical point*, and is special because it is the point that gives rise to the maximum value.) A period four orbit, for instance, might be labelled MRLR, to indicate that starting in the middle, x first goes to the right, then to the left, then to the right, and back to the middle. An alternative period four orbit would be MRLL. It turns out that only certain combinations are allowed; MLRL, for example, is not possible. Furthermore, they showed that as the parameter r is increased, stable periodic orbits appear in a certain manner that can easily be predicted in terms of a simple rule.

The remarkable aspect is that this rule is *universal,* i.e. it is the same for *any* map of the same basic type as the logistic equation. For example, $x_{k+1} = r \sin \pi x_k$ is another map that is similar, but not exactly the same, as the logistic map shown in Figure 1. The ordering of the stable periodic orbits is exactly the same as that of the logistic map, and can be predicted according to the same rule. They called this universal sequence of stable periodic orbits the U-sequence. Although these maps may seem far removed from reality, in fact the existence of the U-sequence has now been verified in experiments on chemical reactors [11].

This concept of universality was extended by Mitchell Feigenbaum [5], who showed that for a certain subset of the U-sequence, not just the order, but also the spacing of the parameter values was the same for all maps of the same basic type as the logistic equation. The subsequence investigated by Feigenbaum, called the period-doubling sequence, is special because it initiates the transition to chaos. What he showed is that number describing the spacing of the parameters for the period-doubling sequence is the same for all maps of this general type. Feigenbaum's predictions have now been verified for many different kinds of physical phenomena.

The existence of the power law behavior given in Equation (2) suggests that the scaling properties are of the right kind to extend Feigenbaum's theory to the whole U-sequence. Furthermore, preliminary results suggest that the number β obtained is the same in each case, although it should be emphasized that as yet these results are inconclusive. These ideas have been worked out in more detail [12] for a similar kind of nonlinear equation, called a circle map, that has another type of transition to chaos.

Building on previous work [13,14], we have shown that the sequence of periodic orbits causing sensitive dependence to parameters lie on a universal strange attractor in parameter space. In other words, there is a geometrical object (an attractor) defining a rule that allows the order and spacing of the periodic orbits to be predicted. The fact that this attractor is *strange* (or chaotic) means that the spacing is very sensitive to the value of parameters. The fact that it is *universal* means that all nonlinear mappings of this type are described by the exactly the same strange attractor. Another approach to different aspects of this same problem has also recently been proposed by Feigenbaum [15]; as yet it is unclear how these two different approaches are related to each other. We believe that our approach can also be extended to understand the U-sequence discussed above.

Conclusions

Thus, even though many nonlinear systems exhibit complicated behavior, we are finding that in many cases there is order underlying this complicated behavior. One example of this is sensitive dependence to parameters, which happens when the parameter values for phenomena of one type (e.g chaos) form a fat fractal. Although on the surface this behavior is very complicated, such fat fractals have well-defined scaling properties. In some cases we have been able to show that there is a sense in which all fractals in a given class are the same, a sense which can be made precise in terms of a universal strange attractor. Such universal strange attractors provide us with a means of classifying behavior, by stating in a precise manner an aspect in which two otherewise different nonlinear equations are alike. The fact that so far only a few fundamentally different type of behavior have been seen, with only a few underlying universal strange attractors, gives hope that it may be possible to group the behavior of nonlinear equations into a finite number of different categories. The universal strange attractor makes explicit the way in which the members of each category are the same. Although phenomena such as sensitive dependence to initial conditions and sensitive dependence to parameters imply limits to prediction, by exploiting the order stemming from their deterministic origins we can at least approach these limits.

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